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On fermionic Novikov algebras

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Abstract

Novikov algebras were introduced in connection with the Poisson brackets of hydrodynamic type and Hamiltonian operators in formal variational calculus. They are a class of left-symmetric algebras with commutative right multiplication operators, which can be viewed as bosonic. Fermionic Novikov algebras are a class of left-symmetric algebras with anti-commutative right multiplication operators. They correspond to a certain Hamiltonian superoperator in a supervariable. In this paper, we commence a study on fermionic Novikov algebras from the algebraic point of view. We will show that any fermionic Novikov algebra in dimension $\leqslant 3$ must be bosonic. Moreover, we give the classification of real fermionic Novikov algebras on four-dimensional nilpotent Lie algebras and some examples in higher dimensions. As a corollary, we obtain kinds of four-dimensional real fermionic Novikov algebras which are not bosonic. All of these examples will serve as a guide for further development including the application in physics.

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1. Introduction

Novikov algebras were introduced in connection with the Poisson brackets of hydrodynamic type [1–3] and Hamiltonian operators in the formal variational calculus [4–9]. A Novikov algebra A is a vector space over a field $\mathbf{F}$ with a bilinear product $(x, y) \to xy$ satisfying

$$(x_1, x_2, x_3) = (x_2, x_1, x_3) \tag{1.1}$$

and

$$(x_1x_2)x_3 = (x_1x_3)x_2 \tag{1.2}$$

for $x_1, x_2, x_3 \in A$, where

$$(x_1, x_2, x_3) = (x_1x_2)x_3 - x_1(x_2x_3). \tag{1.3}$$

Novikov algebras are a special class of left-symmetric algebras which only satisfy equation (1.1). Left-symmetric algebras are a class of non-associative algebras arising from the study of affine manifolds, affine structures and convex homogeneous cones [10–14].

The commutator of a Novikov algebra (or a left-symmetric algebra) A

$$[x, y] = xy - yx \tag{1.4}$$

defines a (sub-adjacent) Lie algebra $\mathcal{G} = \mathcal{G}(A)$. Let L_x and R_x denote the left and right multiplication operators, respectively, i.e. $L_x(y) = xy, R_x(y) = yx, \forall x, y \in A$. Then for a Novikov algebra, the left multiplication operators form a Lie algebra and the right multiplication operators are commutative. A left-symmetric algebra is called right-nilpotent or transitive if every R_x is nilpotent. The transitivity corresponds to the completeness of the affine manifolds in geometry [10, 11].

In fact, the formal variational calculus given by Gel'fand and Dikii in [4, 5] is bosonic, and a fermionic formal variational calculus was given by Xu in [8]. Moreover, motivated by the supersymmetric theory, Xu gave a formal variational calculus of supervariables in [9] which combines the bosonic theory of Gel'fand–Dikii and his fermionic theory. Novikov algebras and certain new algebraic structures are related to the Hamiltonian superoperator in terms of this theory. In particular, the algebraic structure (1.1)–(1.2) can be viewed as a bosonic Novikov algebra due to the commutative right multiplication operators. On the other hand, a fermionic Novikov algebra was introduced as a left-symmetric algebra with anti-commutative right multiplication operators: an algebra is a fermionic Novikov algebra if its product satisfies equation (1.1) and

$$(x_1x_2)x_3 = -(x_1x_3)x_2. \tag{1.5}$$

It corresponds to the following Hamiltonian operator H of type 0 [9]:

$$H^0_{\alpha,\beta} = \sum_{\gamma\in I} \left(a^\gamma_{\alpha,\beta}\Phi_\gamma(2) + b^\gamma_{\alpha,\beta}\Phi_\gamma D\right) \qquad a^\gamma_{\alpha,\beta}, b^\gamma_{\alpha,\beta} \in \mathbb{R}. \tag{1.6}$$

There has been a lot of progress in the study of bosonic Novikov algebras such as the fundamental structure theory of a finite-dimensional bosonic Novikov algebra over an algebraically closed field with characteristic 0 [15], infinite-dimensional simple bosonic Novikov algebras [16–18], finite-dimensional simple bosonic Novikov algebras over an algebraically closed field with prime characteristic [19], the Poisson structures on Novikov algebras [20], the classification of bosonic Novikov algebras in low dimensions [21, 22], the realization of bosonic Novikov algebras [23, 24], the invariant bilinear forms on bosonic Novikov algebras [25], and so on. However, we know very little about fermionic Novikov algebras. One of the reasons is due to very few examples: it is not easy to obtain non-trivial fermionic Novikov algebras, as said in [9]. In fact, a six-dimensional real fermionic Novikov algebra was constructed in [9], which is the first non-bosonic example.

In this paper, we commence to study fermionic Novikov algebras from the algebraic point of view. We mainly give their classification in low dimensions, and some examples in higher dimensions. In particular, we obtain kinds of four-dimensional real fermionic Novikov algebras which are not bosonic. They are the non-bosonic fermionic Novikov algebras in the lowest dimension. Like the study of bosonic Novikov algebras, the study of these examples will serve as a guide for further development.

The paper is organized as follows. In section 2, we give a brief discussion on the general theory of fermionic Novikov algebras. In section 3, we give the classification of fermionic

Novikov algebras over the complex field in dimension 3. We can see that all of them are bosonic. In section 4, based on some results in [10], we give the classification of real fermionic Novikov algebras on four-dimensional nilpotent Lie algebras. In section 5, we obtain some examples in higher dimensions. These examples have an interesting geometric background, although they are both bosonic and fermionic. In section 6, we give some conclusions based on the discussion in the previous sections.

2. Fermionic Novikov algebras

First, we can see that equation (1.5) is equivalent to the following condition:

$$R_x^2 = 0 \qquad \forall x \in A. \tag{2.1}$$

In fact, equation (2.1) can be 'linearized' to equation (1.5) by replacing x by $x + y$. Thus a fermionic Novikov algebra is a left-symmetric algebra the square of whose every right multiplication operator is zero. Hence, we have

Corollary 1. *Every fermionic Novikov algebra is transitive.*

Example 1. The one-dimsional fermionic Novikov algebra A must be trivial, that is, $e_1e_1 = 0$, where e_1 is a basis of A.

Example 2. The classification of left-symmetric algebras in dimension 2 over the complex field was given in [14]. With the additional condition (1.5), it is easy to get the classification of two-dimensional complex fermionic Novikov algebras: recall that the (form) characteristic matrix of a Novikov algebra is defined as

$$\mathcal{A} = \begin{pmatrix} \sum_{k=1}^n c_{11}^k e_k & \cdots & \sum_{k=1}^n c_{1n}^k e_k \\ \cdots & \cdots & \cdots \\ \sum_{k=1}^n c_{n1}^k e_k & \cdots & \sum_{k=1}^n c_{nn}^k e_k \end{pmatrix} \tag{2.2}$$

where $\{e_i\}$ is a basis of A and $e_ie_j = \sum_{k=1}^n c_{ij}^k e_k$. Hence two-dimensional complex fermionic Novikov algebras are

$$\text{(T1)} \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \qquad \text{(T2)} \begin{pmatrix} 0 & 0 \\ 0 & e_1 \end{pmatrix} \qquad \text{(T3)} \begin{pmatrix} 0 & 0 \\ -e_1 & 0 \end{pmatrix}.$$

It is easy to see that the fermionic Novikov algebras appearing in the above examples are bosonic [21]. Let A be a bosonic–fermionic Novikov algebra, then for any $x, y \in A$, we have $R_xR_y = R_yR_x = 0$. That is $(xy)z = 0$ for any $x, y, z \in A$. In particular, a Novikov algebra A is bosonic and fermionic if and only if $L_x = 0$ for any $x \in A^2$, where A^2 is spanned by $ab, a, b \in A$.

Let $\mathcal{R}(A)$ be the Lie algebra generated by $R(A)$, where $R(A)$ is the set spanned by all R_x. Set

$$\mathcal{R}_1 = R(A) \qquad \mathcal{R}_i = [\mathcal{R}_1, \mathcal{R}_{i-1}]. \tag{2.3}$$

Then

$$\mathcal{R}(A) = \mathcal{R}_1 + \cdots + \mathcal{R}_i + \cdots. \tag{2.4}$$

It is the smallest Lie algebra containing $R(A)$.

Claim 1. Let A be a fermionic Novikov algebra. Then

$$\mathcal{R}(A) = \mathcal{R}_1 + \mathcal{R}_2 = R(A) + [R(A), R(A)] = R(A) + R(A)R(A) \tag{2.5}$$

where $R(A)R(A)$ is the subspace spanned by $R_x R_y$. Moreover, $\mathcal{R}(A)$ is a nilpotent Lie algebra.

In fact, for $R_x, R_y \in R(A)$, we have

$$[R_x, R_y] = R_x R_y - R_y R_x = 2R_x R_y \tag{2.6}$$

$$[R_x, R_y R_z] = R_x R_y R_z - R_y R_z R_x = -R_y R_x R_z - R_y R_z R_x = R_y R_z R_x - R_y R_z R_x = 0. \tag{2.7}$$

Hence $\mathcal{R}_n = 0$ for any $n \geqslant 3$. Moreover, for any $a \in \mathcal{R}(A)$, it is easy to show that a is nilpotent. Thus, by Engel's theorem, $\mathcal{R}(A)$ is a nilpotent Lie algebra.

Corollary 2. *Let A be a fermionic Novikov algebra over the complex field. Then there exists* $x \in A, x \neq 0$ *such that* $xy = 0$ *for every* $y \in A$.

In fact, by Engel's theorem, we can choose a basis $\{e_1, \ldots, e_n\}$ in A, such that $R_{e_1}, \ldots, R_{e_n}$ can be put into strictly upper triangular matrices simultaneously under this basis, that is,

$$R_{e_i}(e_j) = \sum_{k=1}^{n} a^i_{jk} e_k \qquad a^i_{jk} = 0 \quad j \geqslant k. \tag{2.8}$$

In particular, $R_{e_i} e_n = 0$ for any e_i. So $e_n y = 0$ for every $y \in A$.

We recall that for a left-symmetric algebra A, the kernel ideal $N(A) = \{a \in A | ax = 0, \forall x \in A\}$ is an ideal of A and $N(A) \neq 0$ if and only if the sub-adjacent Lie algebra contains nontrivial one-parameter subgroups of translations [11]. Thus, by corollary 2, we know

Corollary 3. *The kernel ideal of any finite-dimensional fermionic Novikov algebra over the complex field is non-zero. Hence, there does not exist any finite-dimensional simple fermionic Novikov algebra over the complex field. Here, an algebra is called simple if A does not contain any ideal except zero and itself, and* $A^2 \neq 0$. *Moreover, the sub-adjacent Lie algebra of a fermionic Novikov algebra must contain nontrivial one-parameter subgroups of translations [11].*

At the end of this section, we discuss the Lie transformation algebras of fermionic Novikov algebras: the Lie algebra $\mathcal{L}(A)$ generated by all linear transformations L_x, R_y ($\forall x, y \in A$) is called the Lie multiplication algebra (or Lie transformation algebra). Let $\mathcal{M} = R(A) + L(A)$ denote the set spanned by all L_x, R_y. Then $\mathcal{L}(A)$ is the smallest Lie algebra containing $\mathcal{M}$.

A derivation D of A is a linear transformation satisfying

$$\mathrm{D}(xy) = x\mathrm{D}(y) + \mathrm{D}(x)y \qquad \forall x, y \in A. \tag{2.9}$$

The set $\mathrm{D}(A)$ of derivations is a Lie algebra with the product $[\mathrm{D}_1, \mathrm{D}_2] = \mathrm{D}_1\mathrm{D}_2 - \mathrm{D}_2\mathrm{D}_1$. It is the Lie algebra of the automorphism group of A [26]. A derivation D of A is called an inner derivation if $\mathrm{D} \in \mathcal{L}(A)$. It is easy to see that the set $\mathrm{Inn}(A)$ of all inner derivations is a (Lie) ideal of the Lie algebra $\mathrm{D}(A)$. The inner derivation corresponds the inner automorphism of A. $\mathrm{Inn}(A)$ may also be regarded as a candidate for the space $B^1(A, A)$ of 1-coboundaries. If so, the cohomology $H^1(A, A)$ is just $\mathrm{D}(A)/\mathrm{Inn}\, A$. We have another claim:

Claim 2. The Lie transformation algebra of a fermionic Novikov algebra A is

$$\mathcal{L}(A) = L(A) + R(A) + R(A)R(A). \tag{2.10}$$

In fact, any element of $R(A) + L(A)$ has the form $R_x + L_y$. By equations (1.2), (1.3) and (2.6), we have

$$[L_x, L_y] = L_{[x,y]} \qquad [L_x, R_y] = R_{xy} - R_y R_x \qquad [R_x, R_y] = 2R_x R_y.$$

Therefore

$$\left[L_{x_1}+R_{y_1}, L_{x_2}+R_{y_2}\right]=L_{[x_1,y_1]}+R_{x_1y_2}-R_{y_2}R_{x_1}-R_{x_2y_1}+R_{y_1}R_{x_2}+2R_{y_1}R_{y_2}.$$

Moreover, we have

$$[R_{x_1}R_{x_2}, R_{x_3}]=0$$

and

$$\begin{aligned}\left[R_{x_1}R_{x_2}, L_{x_3}\right]&=R_{x_1}\left[R_{x_2}, L_{x_3}\right]+\left[R_{x_1}, L_{x_3}\right]R_{x_2}\\&=R_{x_1}\left(R_{x_2}R_{x_3}-R_{x_3x_2}\right)+\left(R_{x_1}R_{x_3}-R_{x_3x_1}\right)R_{x_2}\\&=-R_{x_1}R_{x_3x_2}-R_{x_3x_1}R_{x_2}.\end{aligned}$$

Hence $\mathcal{L}(A)=L(A)+R(A)+R(A)R(A)$.

3. Three-dimensional complex fermionic Novikov algebras

In this section, let A be a fermionic Novikov algebra over the complex field in dimension 3. From the discussion in the previous section, we can choose a basis $\{e_1, e_2, e_3\}$ in A such that

$$R_{e_1}=\begin{pmatrix}0 & a_{12}^1 & a_{13}^1\\ 0 & 0 & a_{23}^1\\ 0 & 0 & 0\end{pmatrix}\qquad R_{e_2}=\begin{pmatrix}0 & a_{12}^2 & a_{13}^2\\ 0 & 0 & a_{23}^2\\ 0 & 0 & 0\end{pmatrix}\qquad R_{e_3}=\begin{pmatrix}0 & a_{12}^3 & a_{13}^3\\ 0 & 0 & a_{23}^3\\ 0 & 0 & 0\end{pmatrix}. \tag{3.1}$$

Since $R_{e_i}^2=0$ and $R_{e_i}R_{e_j}+R_{e_j}R_{e_i}=0$, we have

$$a_{12}^i a_{23}^i=0 \qquad i=1,2,3 \tag{3.2}$$

$$a_{12}^i a_{23}^j+a_{12}^j a_{23}^i=0 \qquad i\neq j \quad i,j=1,2,3. \tag{3.3}$$

Claim 3. Any fermionic Novikov algebra over the complex field in dimension 3 must be bosonic.

In fact, we have the following cases:

Case 1: $a_{12}^1\neq 0$. Then by equations (3.2) and (3.3), we have $a_{23}^i=0, i=1,2,3$. Therefore, A satisfies $R_{e_i}R_{e_j}=R_{e_j}R_{e_i}=0$, that is, A is a bosonic Novikov algebra.
Case 2: $a_{23}^1\neq 0$. Then by equations (3.2) and (3.3), we have $a_{12}^i=0$, $i=1,2,3$. Hence, A is a bosonic Novikov algebra since $R_{e_i}R_{e_j}=R_{e_j}R_{e_i}=0$.
Case 3: $a_{12}^1=a_{23}^1=0$. Similar to the discussion in the above cases, we can know that if $a_{12}^2\neq 0$ or $a_{23}^2\neq 0$, A is a bosonic Novikov algebra. So we can only discuss the case $a_{12}^2=a_{23}^2=0$. But, for this case, we still have $R_{e_i}R_{e_j}=R_{e_j}R_{e_i}=0$, hence A is still a bosonic Novikov algebra.

Corollary 4. *Any fermionic Novikov algebra over the real field in dimension $\leqslant 3$ must be bosonic.*

Otherwise, if there exists a real fermionic Novikov algebra which is not bosonic, then its complexification cannot be bosonic, which is contradictory to the above claim.

By the classification of three-dimensional (bosonic) Novikov algebras over the complex field given in [21] and the additional condition $R_{e_i}R_{e_j}=0$, we can easily get the classification of three-dimensional fermionic Novikov algebras over the complex field:

(A1) $\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ (A2) $\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & e_1 \end{pmatrix}$ (A3) $\begin{pmatrix} 0 & 0 & 0 \\ 0 & e_1 & 0 \\ 0 & 0 & e_1 \end{pmatrix}$

(A5) $\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & e_1 \\ 0 & -e_1 & 0 \end{pmatrix}$ (A6) $\begin{pmatrix} 0 & 0 & 0 \\ 0 & e_1 & e_1 \\ 0 & -e_1 & le_1 \end{pmatrix}$ (A8) $\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & e_1 & e_2 \end{pmatrix}$

(A9) $\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & e_2 & 0 \end{pmatrix}$ (A10) $\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & e_2 & e_1 \end{pmatrix}$

(A11) $\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ e_1 & le_2 & 0 \end{pmatrix}$ $(|l| \leqslant 1, l \neq 0)$ (A12) $\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ e_1 & e_1 + e_2 & 0 \end{pmatrix}$.

4. Four-dimensional real fermionic Novikov algebras on nilpotent Lie algebras

It is obvious that the discussion in section 3 cannot be extended to higher dimensions. On the other hand, due to geometric reasons, the study of transitive left-symmetric algebras on nilpotent Lie algebras can be regarded as the first step to constructing a general theory of transitive left-symmetric algebras on all solvable Lie algebras [10]. Hence, Kim gave a classification of transitive left-symmetric algebras on four-dimensional nilpotent Lie algebras over the real field $\mathbb{R}$ through an extension theory in [10]. Based on this result and the additional condition $R_{e_i} R_{e_j} = -R_{e_j} R_{e_i}$, we can get the classification of four-dimensional real fermionic Novikov algebras on nilpotent Lie algebras as follows.

Let $\{e_1, e_2, e_3, e_4\}$ be a basis. There are three four-dimensional nilpotent Lie algebras up to isomorphism:

$$A = \langle e_1, e_2, e_3, e_4 | [e_i, e_j] = 0 \rangle \qquad \text{Abelian}$$
$$H = \langle e_1, e_2, e_3, e_4 | [e_2, e_3] = e_1, \text{ other products are zero} \rangle$$
$$T = \langle e_1, e_2, e_3, e_4 | [e_2, e_3] = e_1, [e_2, e_4] = e_2, \text{ other products are zero} \rangle.$$

On the Lie algebra A, the fermionic Novikov algebras are given as follows (we use the symbols in [10]):

(3) $\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & e_1 & 0 & 0 \\ 0 & 0 & e_1 & 0 \\ 0 & 0 & 0 & e_1 \end{pmatrix}$ (4) $\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & e_1 & 0 & 0 \\ 0 & 0 & e_1 & 0 \\ 0 & 0 & 0 & -e_1 \end{pmatrix}$ $(51)_0$ $\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & e_1 & 0 \\ 0 & 0 & 0 & e_1 \end{pmatrix}$

$(52)_0$ $\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & e_1 & 0 \\ 0 & 0 & 0 & -e_1 \end{pmatrix}$ $(53)_0$ $\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & e_1 & e_2 \\ 0 & 0 & e_2 & -e_1 \end{pmatrix}$ (54) $\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & e_1 & 0 \\ 0 & 0 & 0 & e_2 \end{pmatrix}$

$(57)_0$ $\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & e_1 & e_2 \\ 0 & 0 & e_2 & 0 \end{pmatrix}$ $(60)_1$ $\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & e_1 \\ 0 & 0 & e_1 & e_3 \end{pmatrix}$ (61) $\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & e_1 \end{pmatrix}$

$$(62)\begin{pmatrix}0&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&0\end{pmatrix}.$$

On the Lie algebra H, the fermionic Novikov algebras are given as follows:

$$(5)\begin{pmatrix}0&0&0&0\\0&e_1&e_1&0\\0&-e_1&0&e_1\\0&0&e_1&0\end{pmatrix}\quad(6)\begin{pmatrix}0&0&0&0\\0&0&e_1&0\\0&-e_1&0&e_1\\0&0&e_1&0\end{pmatrix}\quad(7)_t\begin{pmatrix}0&0&0&0\\0&e_1&e_1&0\\0&-e_1&te_1&0\\0&0&0&e_1\end{pmatrix}$$

$$\begin{matrix}(8)_t\\ t\geqslant 0\end{matrix}\begin{pmatrix}0&0&0&0\\0&e_1&e_1&0\\0&0&te_1&0\\0&0&0&-e_1\\&&t\geqslant 0&\end{pmatrix}\quad(30)_0\begin{pmatrix}0&0&0&0\\0&0&0&0\\0&0&e_1&0\\0&e_1&0&e_2\end{pmatrix}\quad(31)_0\begin{pmatrix}0&0&0&0\\0&0&0&0\\0&0&0&e_1\\0&e_1&0&e_2\end{pmatrix}$$

$$(46)\begin{pmatrix}0&0&0&0\\0&0&0&0\\0&0&0&e_2\\0&0&-e_2&0\end{pmatrix}\quad(47)\begin{pmatrix}0&0&0&0\\0&0&0&0\\0&0&e_1&e_1\\0&0&-e_1&0\end{pmatrix}\quad(48)\begin{pmatrix}0&0&0&0\\0&0&0&0\\0&0&e_1&e_2\\0&0&-e_2&0\end{pmatrix}$$

$$(49)\begin{pmatrix}0&0&0&0\\0&0&0&0\\0&0&e_1&e_2\\0&0&-e_2&e_1\end{pmatrix}\quad(50)\begin{pmatrix}0&0&0&0\\0&0&0&0\\0&0&e_1&e_2\\0&0&-e_2&-e_1\end{pmatrix}\quad\begin{matrix}(51)_t\\ t>0\end{matrix}\begin{pmatrix}0&0&0&0\\0&0&0&0\\0&0&e_1&te_1\\0&0&-te_1&e_1\\&&t>0&\end{pmatrix}$$

$$\begin{matrix}(52)_t\\ t>0\end{matrix}\begin{pmatrix}0&0&0&0\\0&0&0&0\\0&0&e_1&te_1\\0&0&-te_1&-e_1\\&&t>0&\end{pmatrix}\quad\begin{matrix}(53)_t\\ t>0\end{matrix}\begin{pmatrix}0&0&0&0\\0&0&0&0\\0&0&e_1&(1+t)e_2\\0&0&(1-t)e_2&-e_1\\&&t>0&\end{pmatrix}$$

$$(55)_t\begin{pmatrix}0&0&0&0\\0&0&0&0\\0&0&e_1&e_1+te_2\\0&0&-e_1-te_2&e_2\end{pmatrix}\quad(56)\begin{pmatrix}0&0&0&0\\0&0&0&0\\0&0&e_1&e_1+e_2\\0&0&-e_1+e_2&0\end{pmatrix}$$

$$\begin{matrix}(57)_t\\ t>0\end{matrix}\begin{pmatrix}0&0&0&0\\0&0&0&0\\0&0&e_1&(1+t)e_2\\0&0&(1-t)e_2&0\\&&t>0&\end{pmatrix}\quad(58)\begin{pmatrix}0&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&e_1&e_3\end{pmatrix}$$

$$(59)\begin{pmatrix}0&0&0&0\\0&0&0&0\\0&0&0&e_1\\0&0&e_2&e_3\end{pmatrix}\quad\begin{matrix}(60)_t\\ t\neq 1\end{matrix}\begin{pmatrix}0&0&0&0\\0&0&0&0\\0&0&0&e_1\\0&0&te_1&e_3\\&&t\neq 1&\end{pmatrix}$$

$$(44)_1 \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & e_1 \\ 0 & 0 & -e_1 & 0 \\ 0 & e_1 & e_2 & e_3 \end{pmatrix}.$$

On the Lie algebra T, the fermionic Novikov algebras are given as follows:

$$(42) \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & e_1 & e_2 & e_3 \end{pmatrix} \qquad \begin{matrix}(44)_t \\ t \neq 1\end{matrix} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & e_1 \\ 0 & 0 & -e_1 & 0 \\ 0 & te_1 & e_2 & e_3 \\ & t \neq 1 & & \end{pmatrix}.$$

Comparing with the classification of (bosonic) transitive Novikov algebras over the real field on four-dimensional nilpotent Lie algebras given in [22], we know that except for the type 44_t (for any $t \in \mathbb{R}$), the fermionic Novikov algebras given above are bosonic. Hence the algebras of type 44_t are kinds of non-bosonic fermionic Novikov algebras in the lowest dimension.

5. Some fermionic Novikov algebras in higher dimensions

From the discussion in [11], we know that a left-invariant connection ∇ on G is adapted to the automorphism structure of G if and only if the linear mapping $\theta : \mathcal{G} \to gl(\mathcal{G})$ defined by $\theta(x) = \nabla_x$ takes values in the algebra $\mathrm{Der}(\mathcal{G})$, where $\mathrm{Der}(\mathcal{G})$ is the Lie algebra of the derivations of the Lie algebra $\mathcal{G}$. Hence we call a left-symmetric algebra (a bosonic or fermionic Novikov algebra) A a derivation algebra if its every left multiplication operator L_x or its every right multiplication operator R_x is a derivation of its sub-adjacent Lie algebra $\mathcal{G}(A)$. Therefore, the Lie group G possesses a left-invariant locally flat connection defined by a left-symmetric algebra (a bosonic or fermionic Novikov algebra) which is adapted to the structure of its automorphisms if and only if the Lie algebra $\mathcal{G}$ is sub-adjacent to a left-symmetric derivation algebra (a bosonic or fermionic Novikov derivation algebra).

It is quite interesting to see that many examples of left-symmetric derivation algebras given in [11] are both bosonic [27] and fermionic Novikov algebras. Thus, in this section, we obtain some fermionic Novikov algebras in higher dimensions, although they are also bosonic. We would like to point out that these example are still important and interesting due to their geometric background given above. For the paper to be self-contained, we briefly introduce these examples as follows.

Example 3. There are two important classes of fermionic Novikov derivation algebras in dimension 5 given in [11] with the following characteristic matrices, respectively:

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ -e_1 & -e_2 & 0 & e_1 & e_1 \\ -e_1 & -e_1 & e_1 - e_2 & \frac{1}{2}(e_1+e_2) & \frac{1}{2}(e_1+e_2) \\ -e_1 & -e_1 & e_1 - e_2 & \frac{1}{2}(e_1+e_2) & \frac{1}{2}(e_1+e_2) \end{pmatrix}$$

$$\begin{pmatrix} \lambda e_3 + \beta e_4 + \mu e_5 & e_3 + \lambda e_4 + \gamma e_5 & e_4 & e_5 & 0 \\ \lambda e_4 + \gamma e_5 & e_4 + \delta e_5 & e_5 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Example 4. We can construct a series of fermionic Novikov derivation algebras in dimension $\geqslant 5$ through the extension of a five-dimensional fermionic Novikov derivation algebra: let A be the Lie algebra in dimension 5 with the following non-zero products:

$$[e_1, e_3] = e_5 \qquad [e_1, e_3] = e_3 \qquad [e_2, e_4] = e_4.$$

A fermionic Novikov derivation product on A is obtained by taking for the left multiplication operators the following endomorphisms:

$$L_{e_1} = \mathrm{ad}(e_1) \qquad L_{e_2} = \mathrm{ad}^2(e_2) \qquad L_{e_3} = L_{e_4} = L_{e_5} = 0,$$

where ad is the adjoint operator of Lie algebra, that is, $\mathrm{ad}(x)(y) = [x, y]$. Consider the Lie algebra $A' = A \times \mathbf{C}e_6$ obtained from A by imposing

$$[e_1, e_5] = e_6 \qquad [e_i, e_6] = 0 \qquad \text{for} \quad 1 \leqslant i \leqslant 6.$$

The fermionic Novikov derivation product on A' is given as

$$L'_{e_1} = \mathrm{ad}'(e_1) \qquad L'_{e_2} = \mathrm{ad}'^2(e_2) \qquad L'_{e_3} = L'_{e_4} = L'_{e_5} = L'_{e_6} = 0.$$

Thus, by a series of such extensions we can obtain a series of fermionic Novikov derivation algebras.

Example 5. Let A be a 2-solvable Lie algebra, that is, the derived ideal $[A, A]$ is Abelian. Suppose A can be decomposed as a direct sum of subspaces $A = \mathcal{D}(A) \oplus S$ with $[S, S] \subset C(A)$, where $\mathcal{D}(A) = [A, A]$ and $C(A)$ is the centre of A. For every element a in A, we denote by $a_{\mathcal{D}}$ and a_S the respective components of a in $\mathcal{D}(A)$ and S. Then

$$ab = \left[a_S, b_{\mathcal{D}} + \tfrac{1}{2} b_S\right]$$

defines a fermionic Novikov derivation product on A.

Example 6. There exists a fermionic Novikov derivation product on any 2-solvable Lie algebra with trivial centre. In fact, from the discussion in [11], such a Lie algebra A has a decomposition:

$$A = \mathcal{D}(A) \oplus C$$

where C is an Abelian Cartan subalgebra of A. Then A satisfies the condition in example 5 since $[C, C] = \{0\} = C(A)$. Thus the fermionic Novikov derivation product on A can be defined by

$$L_{a_{\mathcal{D}}} = 0 \qquad L_{a_C} = \mathrm{ad}(a_C)$$

where $a_{\mathcal{D}} \in \mathcal{D}, a_C \in C$.

Example 7. There are certain kinds of 2-solvable Lie algebras with the trivial centre having the property that it is sub-adjacent to a unique fermionic Novikov derivation structure. Such an example can be obtained from [11]: let A be an n-dimensional Lie algebra with the product

$$[e_i, e_j] = 0 \quad i, j \geqslant 2 \qquad [e_1, e_i] = \lambda_i e_i \quad i \geqslant 2 \quad \lambda_i \neq 0, \text{the } \lambda_i \text{ being pairwise distinct.}$$

The (unique) fermionic Novikov derivation structure is given by

$$e_1 e_1 = 0 \qquad e_1 e_i = \lambda_i e_i \qquad e_i e_j = 0 \quad i, j \geqslant 2.$$

Example 8. A filiform Lie algebra A in dimension n is an $(n-1)$-step nilpotent Lie algebra, that is, the lower central series $\{A^k\}$ of A ($A^0 = A$ and $A^k = [A^{k-1}, A]$ for $k \geqslant 1$) satisfying $A^{n-1} = 0$, $A^{n-2} \neq 0$. The study of filiform Lie algebra is quite important [28]. For example, the first example of the nilpotent Lie algebra which is not sub-adjacent to a left-symmetric algebra is a filiform Lie algebra [29]. Let A be a filiform Lie algebra with an Abelian commutator subalgebra. Then the product is given by [28] (non-zero products)

$$[e_1, e_i] = e_{i+1} \quad i = 2, \ldots, n-1 \qquad [e_2, e_i] = \sum_{k=i+2}^{n} \alpha_{2,k-i+3} e_k \quad i = 3, \ldots, n-2$$

with parameters $\alpha_{2,s}$, where $5 \leqslant s \leqslant n$. Then it is easy to check that the algebra given by the following products is a fermionic Novikov derivation algebra:

$$\begin{array}{ll} e_1 e_i = e_{i+1} \quad i = 2, \ldots, n-1 & e_2 e_i = [e_2, e_i] \quad i = 3, \ldots, n-2 \\ e_2 e_2 = \alpha_{2,5} e_4 + \cdots + \alpha_{2,n} e_{n-1} & e_i e_j = 0 \quad \text{otherwise.} \end{array}$$

6. Conclusions and discussion

According to the discussion in previous sections, we have the following conclusions:

(1) There does not exist any finite-dimensional simple fermionic Novikov algebra over the complex field since its kernel ideal is not zero. However, it is still unknown whether it is true for an infinite-dimensional fermionic Novikov algebra or over other fields.
(2) All fermionic Novikov algebras in dimension $\leqslant 3$ over the complex or real field are bosonic and there exist kinds of four-dimensional non-bosonic fermionic Novikov algebras. Hence these four-dimensional examples are in the lowest dimension.
(3) It is interesting to see that many left-symmetric derivation algebras are both bosonic and fermionic Novikov algebras, that is, for these examples of left-symmetric algebras, the conditions of a derivation algebra, a bosonic Novikov algebra and a fermionic Novikov algebra are consistent.

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